A K-THEORETIC CLASSIFICATION FOR CERTAIN $\mathbf{Z}/p\mathbf{Z}$ ACTIONS ON AF ALGEBRAS

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Abstract. A K-theoretic classification is given of the C*-dynamical systems $\varinjlim(A_n, \alpha_n, G)$ where A_n is finite dimensional and G is any cyclic group of prime order (namely, $G = Z/pZ$ for some prime number p). Such actions contain N. C. Phillips' natural examples of finite group actions on UHF algebras which don't have the tracial Rokhlin property in [8].

1. Introduction

A number of results concerning the classification of C*-algebras have been obtained under the Elliott programme. However, classification of group actions on C*-algebras is still a far less developed subject, partially because of K-theoretical difficulties. When the C*-algebra and the group action have an inductive limit structure, then the equivariant version of Elliott's intertwining argument can be used for classifying such group actions.

Given a compact group G, let $A = \varinjlim A_n$ be the inductive limit of a sequence of finite dimensional C*-algebras, let $\alpha = \varinjlim \alpha_n$ be an inductive limit action of G on A. Then one can form the C^*-algebra cross product $A \rtimes_\alpha G = \varinjlim A_n \rtimes_{\alpha_n} G$. If each α_n is given by inner automorphisms arisen from a unitary representation of the group G, then it was shown in [5] that the natural K-theory data of $A \rtimes_\alpha G$ is a complete invariant for the C*-dynamical system (A, α, G). Such actions they referred to as locally representable. In the case that A is unital, the K-theory data in [5] consists of the K-group $\mathrm{K}_0(A \rtimes_\alpha G)$ together with (i) the natural order structure, (ii) the special element coming from the projection given by averaging the canonical unitaries of the cross product, (iii) the natural module structure over the representation ring $\mathrm{K}_0(G)$. In [7], Kishimoto considered actions of finite groups on inductive limit algebras with more complicated building blocks (circles), and in [2], this study was extended to still more complicated inductive limit systems and to general compact groups.

In all of these cases above, it was assumed that the actions still satisfied a local representability condition. So it is interesting to consider the case in which the group action is not necessarily inner. Along this line, in [4], G. A. Elliott and H. Su removed this local representability hypothesis in the case where the group is $Z/2Z$ and the building blocks are finite dimensional. In [10], this local representability condition was also removed, where the group is still $Z/2Z$, but the inductive limits are certain real rank zero systems built on some subhomogeneous graph C*-algebras. Then, it is an natural question that to which extent one can

2000 *Mathematics Subject Classification.* Primary 46L35, Secondary 46L55.

Key words and phrases. C*-dynamical system, Classification.

obtain a K-theoretic classification for more general group actions on C*-algebras, conceivably, finite abelian group actions will be a quite large class. To do this, from group structure theory, the p (prime) groups (groups with order being some power of p) cases will be fundamental, and among them, the cyclic groups with prime orders should be the first test case. In the present paper, a K-theoretic classification for inductive limit actions of cyclic groups with prime orders on AF (approximately finite dimensional) algebras will be obtained.

On the other hand, there is another class of group actions on C*-algebras which draw many people's attention, namely, the group actions with (tracial) Rokhlin property. For the integer group Z and Z^d, I. Hirshberg, W. Winter, and J. Zacharias (in [6]) and N. C. Phillips (in [9]) showed that the actions with the tracial Rokhlin property are generic for nice C*-algebras (for example, TAF algebras). For finite group action case, N. C. Phillips give natural examples in [8] that there are inductive limit actions of cyclic groups on UHF algebras which don't have the tracial Rokhlin property. We quote his example of Z_2 action here:

$$\mathcal{A} = \varinjlim \mathrm{M}_{2^n},\ \alpha = \varinjlim \alpha_n,\ \alpha_n = Ad \begin{pmatrix} 1_{2^n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

where the distribution of the eigenvalues of the unitaries indicate that this action doesn't have tracial Rokhlin property (see **Example 2.9** in [8] for detail). Since such examples have inductive limit structure, they sit in our classification results.

Throughout this paper, let us denote the group Z/pZ by Z_p, where p is a prime. To state the invariant, let A be a unital C*-algebra, and let α be a group action of Z_p on A. The invariant we need is as follows:

(1) $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A], \alpha_*)$,

(2) $(\mathrm{K}_0(A \rtimes_\alpha Z_p), \mathrm{K}_0(A \rtimes_\alpha Z_p)^+, \zeta, \hat{\alpha}_*)$, where ζ is the special element in $\mathrm{K}_0(A \rtimes_\alpha Z_p)$ and $\hat{\alpha}$ is the dual action of $\hat{Z}_p$ on $A \rtimes_\alpha Z_p$,

(3) $\iota_* : \mathrm{K}_0(A) \to \mathrm{K}_0(A \rtimes_\alpha Z_p)$, where ι is the canonical embedding of A into $A \rtimes_\alpha Z_p$.

(1) and (3) are necessary, since the action may not be inner, the information in $\mathrm{K}_0(A)$ may not be recovered completely from $\mathrm{K}_0(A \rtimes_\alpha Z_p)$, we must adjoin this, as well as the actions on the K-groups, to the invariant. We state the main theorem here.

Theorem 1.1. *Let $(A, \alpha, Z_p) = \varinjlim (A_n, \alpha_n, Z_p)$ and $(B, \beta, Z_p) = \varinjlim (B_n, \beta_n, Z_p)$ be two approximately finite dimensional inductive limit C*-dynamical systems, let F be an scaled order preserving group isomorphism from $(K_0(A), \alpha_*)$ to $(K_0(B), \beta_*)$, and let ϕ be an order preserving group isomorphism from $(K_0(A \rtimes_\alpha Z_p), \hat{\alpha}_*)$ to $(K_0(B \rtimes_\beta Z_p), \hat{\beta}_*)$ mapping the special element to the special element. Suppose that the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha Z_p) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \phi} \\ K_0(B) & \longrightarrow & K_0(B \rtimes_\beta Z_p). \end{array}$$

Then there is an isomorphism ψ from (A, α, Z_p) to (B, β, Z_p) such that $\psi_* = F$ and such that the extension of ψ to $A \rtimes_\alpha Z_p$ induces ϕ.

The paper is organized as follows. In Section 2, some preliminaries are given about the crossed products of finite dimensional C*-algebras with Z_p actions. In Section 3, an existence result is proved, namely, morphisms between the invariant of the finite dimensional C*-dynamical systems can be lifted to morphisms between the finite dimensional C*-dynamical systems. In Section 4, a uniqueness result is obtained, namely, for any two morphisms between the finite dimensional C*-dynamical systems, if their induced maps agree on the invariant, then they are unitarily equivalent by an equivariant unitary, i.e., a unitary in the fixed point subalgebra of the codomain algebra. These two results are the main ingredients in Elliott's intertwining argument. In Section 5, the main theorem will be proved by Elliott's intertwining argument.

2. PRELIMINARIES

Let $A = \bigoplus_{k=1}^{m} \mathrm{M}_{n_k}$ be a finite dimensional C*-algebra, and let α be a group action of Z_p on A. Since Z_p is cyclic, then α is determined by the corresponding automorphism of the generator of Z_p. Let σ be an order p automorphism of A. From basic representation theory, σ can be decomposed into a finite direct sum of irreducible actions. Each irreducible action has the form either (M_n, ρ) or $(\underbrace{\mathrm{M}_n \oplus ... \oplus \mathrm{M}_n}_{p}, \rho)$. Let us prepare all the K-theoretic information about the irreducible actions.

In the case (M_n, ρ), ρ is given by an order p unitary $V \in \mathrm{M}_n$:

$$\rho(a) = VaV^*,\ a \in \mathrm{M}_n,$$

V could be chosen to be diagonal.

Lemma 2.1. *$M_n \rtimes_\alpha Z_p$ is isomorphic to $\underbrace{M_n \oplus ... \oplus M_n}_{p}$.*

Proof. The identification map is given as follows:

$$\begin{aligned} &a_0 + a_1U_\rho + a_2U_{\rho^2} + ... + a_{p-1}U_{\rho^{p-1}} \\ &\qquad \longrightarrow (a_0 + a_1V + a_2V^2 ... + a_{p-1}V^{p-1}, \\ &\qquad a_0 + e^{i\frac{2\pi}{p}}a_1V + e^{i\frac{4\pi}{p}}a_2V^2 + ... + e^{i\frac{2(p-1)\pi}{p}}a_{p-1}V^{p-1}, \\ &\qquad a_0 + e^{i\frac{4\pi}{p}}a_1V + e^{i\frac{8\pi}{p}}a_2V^2 + ... + e^{i\frac{2(p-2)\pi}{p}}a_{p-1}V^{p-1}, \\ &\qquad ,..., \\ &\qquad a_0 + e^{i\frac{2(p-1)\pi}{p}}a_1V + e^{i\frac{2(p-2)\pi}{p}}a_2V^2 + ... + e^{i\frac{2\pi}{p}}a_{p-1}V^{p-1}), \end{aligned}$$

where $U_{\rho^k}, k = 1, ..., p-1$ are the canonical unitaries in the cross product algebra. Then one can verify the lemma by this formula. $\square$

Remark 2.2. This lemma is also true for non prime p.

Then $\mathrm{K}_0(\mathrm{M}_n) = Z$ and $\mathrm{K}_0(\mathrm{M}_n \rtimes Z_p) = \underbrace{Z \oplus ... \oplus Z}_{p}$, and the map from $\mathrm{K}_0(\mathrm{M}_n)$ to $\mathrm{K}_0(\mathrm{M}_n \rtimes Z_p)$ sends x to $(\underbrace{x, ..., x}_{p})$ and ρ_* is trivial. It is well known that $\hat{Z}_p = Z_p$,

and the generator of $\hat{Z}_p$ is $\hat{\rho}$ which takes the identity element to 1, and takes ρ to $e^{i\frac{2\pi}{p}}$. So

$$\hat{\rho}(\sum_{k=0}^{p-1} a_k U_{\rho^k}) = a_0 + e^{-i\frac{2\pi}{p}} a_1 U_\rho + e^{-i\frac{4\pi}{p}} a_2 U_{\rho^2} + ... + e^{-i\frac{2(p-1)\pi}{p}} a_{p-1} U_{\rho^{p-1}}$$

by the identification formula above, it is easily to see that $\hat{\rho}$ and $\hat{\rho}_*$ is the permutation given by $(\xi_1, \xi_2, ..., \xi_p) \to (\xi_p, \xi_1, ..., \xi_{p-1})$. Take the special element $\zeta = (l_0, ..., l_{p-1})$, where l_k is the number of the eigenvalue $e^{i\frac{2k\pi}{p}}$ of the unitary V which implements the automorphism of ρ.

In the case $(\underbrace{\mathrm{M}_n \oplus ... \oplus \mathrm{M}_n}_{p}, \rho)$, up to conjugacy, ρ can be chosen to have the form: $\rho(a_1, a_2, ..., a_p) = (a_p, a_1, ..., a_{p-1})$.

Lemma 2.3. *$\underbrace{M_n \oplus ... \oplus M_n}_{p} \rtimes_\alpha Z_p$ is isomorphic to* M_{pn}.

Proof. The identification map is given as follows:

$$(a_0^0, a_1^0, ..., a_{p-1}^0) + (a_0^1, a_1^1, ..., a_{p-1}^1) U_\rho + ... + (a_0^{p-1}, a_1^{p-1}, ..., a_{p-1}^{p-1}) U_{\rho^{p-1}}$$
$$\longrightarrow \begin{pmatrix} a_0^0 & a_0^1 & \cdots & a_0^{p-1} \\ a_{p-1}^{p-1} & a_{p-1}^0 & \cdots & a_{p-1}^{p-2} \\ \vdots & \vdots & & \vdots \\ a_1^1 & a_1^2 & \cdots & a_1^0 \end{pmatrix}.$$

□

Then $\mathrm{K}_0(\underbrace{\mathrm{M}_n \oplus ... \oplus \mathrm{M}_n}_{p} \rtimes_\alpha Z_p) = Z$, the canonical map between the K-groups sends $(x_1, ..., x_p)$ to $\sum_{k=1}^{p} x_k$. Let

$$\xi = (a_0^0, a_1^0, ..., a_{p-1}^0) + (a_0^1, a_1^1, ..., a_{p-1}^1) U_\rho + ... + (a_0^{p-1}, a_1^{p-1}, ..., a_{p-1}^{p-1}) U_{\rho^{p-1}},$$

so

$$\begin{aligned} \hat{\rho}(\xi) = & (a_0^0, a_1^0, ..., a_{p-1}^0) + e^{-i\frac{2\pi}{p}} (a_0^1, a_1^1, ..., a_{p-1}^1) U_\rho + ... \\ & + e^{-i\frac{2(p-1)\pi}{p}} (a_0^{p-1}, a_1^{p-1}, ..., a_{p-1}^{p-1}) U_{\rho^{p-1}}. \end{aligned}$$

Then by the identification in Lemma 2.2, the dual action is as follows:

$$\hat{\rho}(C) = \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{p}} & & \\ & & \ddots & \\ & & & e^{i\frac{2(p-1)\pi}{p}} \end{pmatrix} C \begin{pmatrix} 1 & & & \\ & e^{-i\frac{2\pi}{p}} & & \\ & & \ddots & \\ & & & e^{-i\frac{2(p-1)\pi}{p}} \end{pmatrix},$$

for all $C \in M_{pn}$. Hence ρ_* is the permutation and $\hat{\rho}_*$ is trivial. Take the special element ζ to be n.

Remark 2.4. Lemma 2.3 also verify the Takai Duality for (M_n, Z_p).

3. EXISTENCE

In this section, we are going to establish an existence result, which states that morphisms between the invariant of the finite dimensional C*-dynamical systems can be lifted to morphisms between the finite dimensional C*-dynamical systems. This existence theorem together with the uniqueness theorem in next section are the two main ingredients in Elliott's intertwining argument.

Theorem 3.1. *Let (A_k, α_k, Z_p) and (B_n, β_n, Z_p) be two irreducible finite dimensional C*-dynamical systems. Let F_k be an ordered group morphism from $(K_0(A_k), [1_{A_k}], \alpha_{k*})$ to $(K_0(B_n), [1_{B_n}], \beta_{n*})$. Let ϕ_k be an ordered group morphism from $(K_0(A_k \rtimes_{\alpha_k} Z_p), \hat{\alpha}_{k*})$ to $(K_0(B_n \rtimes_{\beta_n} Z_p), \hat{\beta}_{n*})$, which preserves the special element. Then there exists a homomorphism ψ_k from (A_k, α_k, Z_P) to (B_n, β_n, Z_p), such that $\psi_{k*} = F_k$, and $\widetilde{\psi}_{k*} = \phi_k$, where $\widetilde{\psi}_k$ is the natural extension of ψ_k to $A_k \rtimes_{\alpha_k} Z_p$.*

Proof. We are going to prove the theorem in four different cases. Assume $Z_p = \{\rho, \rho^2, \ldots, \rho^{p-1}, 1\}$.

(1). $A_k = M_k,\ B_n = M_n$.

Suppose $U \in M_k$ and $V \in M_n$ are the two unitaries which implement the automorphism for ρ on M_k and M_n respectively. Since F_k preserves the scale, then $F_k = \frac{n}{k}$. By Lemma 2.1, ϕ_k is of the form:

$$\phi_k = \begin{pmatrix} l_{11} & l_{12} & \ldots & l_{1p} \\ l_{21} & l_{22} & \ldots & l_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ l_{p1} & l_{p2} & \ldots & l_{pp} \end{pmatrix}.$$

Moreover, ϕ_k intertwines $\hat{\alpha}_{k*}$ and $\hat{\beta}_{n*}$, by calculation, one obtains that

$$\phi_k = \begin{pmatrix} l_{11} & l_{12} & \ldots & l_{1p} \\ l_{1p} & l_{11} & \ldots & l_{1p-1} \\ \ldots & \ldots & \ldots & \ldots \\ l_{12} & l_{13} & \ldots & l_{11} \end{pmatrix}.$$

By assumption, we have $\phi_k\zeta = \zeta'$, where ζ and ζ' are the two special elements in $M_k \rtimes_{\alpha_k} Z_p$ and $M_n \rtimes_{\beta_n} Z_p$, then $(l_{11} + l_{12} + \ldots + l_{1p})k = n$.

Define

$$\begin{aligned} &e_1 = e_2 = \ldots = e_{l_{11}} = I_k, \\ &e_{l_{11}+1} = e_{l_{11}+2} = \ldots = e_{l_{11}+l_{12}} = e^{-i\frac{2\pi}{p}} \otimes id_{\frac{n}{k}}, \\ &e_{l_{11}+l_{12}+1} = e_{l_{11}+l_{12}+2} = \ldots = e_{l_{11}+l_{12}+l_{13}} = e^{-i\frac{4\pi}{p}} \otimes id_{\frac{n}{k}}, \\ &, \ldots, \end{aligned}$$

set

$$e = diag(e_1, \ldots, e_{l_{11}}, e_{l_{11}+1}, \ldots, e_{l_{11}+l_{12}}, \ldots, e_{l_{11}+\ldots+l_{1p}}),$$

then $e(U \otimes id_{\frac{n}{k}}) = (U \otimes id_{\frac{n}{k}})e$.

Because ϕ_k preserves the special elements (and the choice of special element in this case), then the eigenvalue list of $(U \otimes id_{\frac{n}{k}})e$ is the same as V, then there exists a unitary W, such that $W^*VW = (U \otimes id_{\frac{n}{k}})e$.

Define a homomorphism $\psi_k : M_k \to M_n$ by:

$$\psi_k(a) = W(a \otimes id_{\frac{n}{k}})W^*,$$

then $(U^* \otimes id_{\frac{n}{k}})W^*VW(a \otimes id_{\frac{n}{k}}) = (a \otimes id_{\frac{n}{k}})(U^* \otimes id_{\frac{n}{k}})W^*VW$,
namely, ψ_k intertwines α_k and β_n, and $\psi_{k*} = F_k$. Since the natural extension $\widetilde{\psi}_k$ intertwines $\hat{\alpha}_k$ and $\hat{\beta}_n$, by calculation, $\widetilde{\psi}_{k*} = \phi_k$.

(2). $A_k = M_k,\ B_n = \underbrace{M_n \oplus ... \oplus M_n}_{p}$.

Obviously, $F_k = \begin{pmatrix} \frac{n}{k} \\ \vdots \\ \frac{n}{k} \end{pmatrix}$. Since ϕ_k intertwines $\hat{\alpha}_{k*}$ and $\hat{\beta}_{n*}$, by calculation, we have: $\phi_k = \begin{pmatrix} l_1 & l_1 & \cdots & l_1 \end{pmatrix}$. Moreover, because ϕ_k preserves the special element, then $l_1 = \frac{n}{k}$. Let V be the unitary implementing the automorphism for ρ on M_k.

Define a homomorphism $\psi_k : M_k \to \underbrace{M_n \oplus ... \oplus M_n}_{p}$ by:

$$\psi_k(a) = (W_1(a \otimes id_{\frac{n}{k}})W_1^*, W_2(a \otimes id_{\frac{n}{k}})W_2^*, ..., W_p(a \otimes id_{\frac{n}{k}})W_p^*),$$

where

$$W_1 = 1 \otimes id_{\frac{n}{k}},\ W_2 = V^* \otimes id_{\frac{n}{k}},\ ...,\ W_p = (V^*)^{p-1} \otimes id_{\frac{n}{k}}.$$

Then it is easily to check that ψ_k intertwines α_k and β_n, and $\psi_{k*} = F_k$. The natural extension $\widetilde{\psi}_k$ intertwines $\hat{\alpha}_k$ and $\hat{\beta}_n$, so $\widetilde{\psi}_{k*} = \phi_k$.

(3). $A_k = \underbrace{M_k \oplus ... \oplus M_k}_{p},\ B_n = M_n$.

Since F_k intertwines α_k and β_n, by calculation, $F_k = \begin{pmatrix} \frac{n}{pk} & \frac{n}{pk} & \cdots & \frac{n}{pk} \end{pmatrix}$. Since ϕ_k intertwines $\hat{\alpha}_{k*}$ and $\hat{\beta}_{n*}$, we have $\begin{pmatrix} l_1 \\ l_1 \\ \vdots \\ l_1 \end{pmatrix}$, moreover by the assumption on the special elements, $l_1 = \frac{n}{pk}$. Let V be the unitary implementing the automorphism for ρ on M_n. To define a homomorphism which intertwines α_k and β_n, we need to find a unitary W, such that $\mathrm{Ad}(W^*VW)$ maps $\mathrm{diag}(a_1 \otimes id_{\frac{n}{pk}}, a_2 \otimes id_{\frac{n}{pk}}, ..., a_p \otimes id_{\frac{n}{pk}})$ to $\mathrm{diag}(a_p \otimes id_{\frac{n}{pk}}, a_1 \otimes id_{\frac{n}{pk}}, ..., a_{p-1} \otimes id_{\frac{n}{pk}})$ for all $(a_1, ..., a_p)$ in $\underbrace{M_k \oplus ... \oplus M_k}_{p}$. By Lemma IV.2 in [5], this can be done.

(4). $A_k = \underbrace{M_k \oplus ... \oplus M_k}_{p},\ B_n = \underbrace{M_n \oplus ... \oplus M_n}_{p}$.

Since F_k intertwines α_k and β_n, by calculation, we have:

$$F_k = \begin{pmatrix} l_{11} & l_{12} & \dots & l_{1p} \\ l_{1p} & l_{11} & \dots & l_{1p-1} \\ \cdots & \cdots & \cdots & \cdots \\ l_{12} & l_{13} & \dots & l_{11} \end{pmatrix},$$

and $(l_{11} + l_{12} + ... + l_{1p})k = n$. Similarly, we also have $\phi_k = \frac{n}{k}$.

Define a homomorphism $\psi_k : \underbrace{M_k \oplus ... \oplus M_k}_{p} \to \underbrace{M_n \oplus ... \oplus M_n}_{p}$ by:

$$\psi_k(a_1, a_2, ..., a_p) = (\psi_k^1(\cdot), \psi_k^2(\cdot), ..., \psi_k^p(\cdot)),$$

where $(\cdot)$ is the abbreviation of $(a_1, a_2, ..., a_p) \in \underbrace{M_n \oplus ... \oplus M_n}_{p}$, and

$$\begin{aligned}
\psi_k^1(a_1, a_2, ..., a_p) =&\mathrm{diag}(a_1 \otimes id_{l_{11}}, a_2 \otimes id_{l_{12}}, ..., a_p \otimes id_{l_{1p}}),\\
\psi_k^2(a_1, a_2, ..., a_p) =&\mathrm{diag}(a_2 \otimes id_{l_{11}}, a_3 \otimes id_{l_{12}}, ..., a_1 \otimes id_{l_{1p}}),\\
\psi_k^3(a_1, a_2, ..., a_p) =&\mathrm{diag}(a_3 \otimes id_{l_{11}}, a_4 \otimes id_{l_{12}}, ..., a_2 \otimes id_{l_{1p}}),\\
, ... ,&\\
\psi_k^p(a_1, a_2, ..., a_p) =&\mathrm{diag}(a_p \otimes id_{l_{11}}, a_1 \otimes id_{l_{12}}, ..., a_{p-1} \otimes id_{l_{1p}}).
\end{aligned}$$

Then ψ_k satisfies the condition. □

Corollary 3.2. *Let (A_k, α_k, Z_p) and (B_n, β_n, Z_p) be two finite dimensional C*-dynamical systems. Let F_k be an ordered group morphism from $(K_0(A_k), [1_{A_k}], \alpha_{k*})$ to $(K_0(B_n), [1_{B_n}], \beta_{n*})$. Let ϕ_k be an ordered group morphism from $(K_0(A_k \rtimes_{\alpha_k} Z_p), \hat{\alpha}_{k*})$ to $(K_0(B_n \rtimes_{\beta_n} Z_p), \hat{\beta}_{n*})$, which preserves the special element, and the following diagram*

$$\begin{array}{ccc}
K_0(A_k) & \longrightarrow & K_0(A_k \rtimes_{\alpha_k} Z_p)\\
\Big\downarrow F_k & & \Big\downarrow \phi_k\\
K_0(B_n) & \longrightarrow & K_0(B_n \rtimes_{\beta_n} Z_p)
\end{array}$$

commutes. Then there exists a homomorphism ψ_k from (A_k, α_k, Z_p) to (B_n, β_n, Z_p), such that $\psi_{k*} = F_k$, and $\widetilde{\psi}_{k*} = \phi_k$, where $\widetilde{\psi}_k$ is the natural extension of ψ_k to $A_k \rtimes_{\alpha_k} Z_p$.

Proof. □

4. UNIQUENESS

In this section, we are going to establish the uniqueness theorem, namely, if two morphisms between the finite dimensional C*-dynamical systems agree on the K-theoretic invariants, then they are unitarily equivalent by an equivariant unitary, namely, an unitary in the fixed point subalgebra of the codomain algebra.

Theorem 4.1. *Let ϕ_k and ψ_k be two homomorphisms from the irreducible finite dimensional C*-dynamical system (A_k, α_k, Z_p) to (B_n, β_n, Z_p). Denote by $\widetilde{\phi}_k$ and $\widetilde{\psi}_k$ the morphisms from $A_k \rtimes_{\alpha_k} Z_p$ to $B_n \rtimes_{\beta_n} Z_p$ induced by ϕ_k and ψ_k, respectively. If $\phi_{k*} = \psi_{k*}$ and $\widetilde{\phi}_{k*} = \widetilde{\psi}_{k*}$, then there exists a unitary W in $B_n^{\beta_n}$, the fixed point subalgebra of B_n, such that $\phi_k = AdW \circ \psi_k$.*

Proof. Again we are going to prove the theorem in four cases. Assume $Z_p = \{\rho, \rho^2, ..., \rho^{p-1}, 1\}$.

(1). $A_k = \mathrm{M}_k, B_n = \mathrm{M}_n$.

Let $U \in \mathrm{M}_k$ and $V \in \mathrm{M}_n$ be the two unitaries which implement the action α_k and β_n, respectively. Let X and Y be two unitaries in B_n such that

$$\phi(a) = X(a \otimes id_{\frac{n}{k}})X^*, \psi(a) = Y(a \otimes id_{\frac{n}{k}})Y^*, \forall a \in \mathrm{M}_k.$$

Since both ϕ and ψ intertwine the actions α_k and β_n, we have that

$$X(UaU^* \otimes id_{\frac{n}{k}})X^* = VX(a \otimes id_{\frac{n}{k}})X^*V^*,$$
$$Y(UaU^* \otimes id_{\frac{n}{k}})Y^* = VY(a \otimes id_{\frac{n}{k}})Y^*V^*.$$

Hence,

$$\begin{aligned}X^*V^*X(U\otimes id_{\frac{n}{k}})(a\otimes id_{\frac{n}{k}}) &= (a\otimes id_{\frac{n}{k}})X^*V^*X(U\otimes id_{\frac{n}{k}}),\\ Y^*V^*Y(U\otimes id_{\frac{n}{k}})(a\otimes id_{\frac{n}{k}}) &= (a\otimes id_{\frac{n}{k}})Y^*V^*Y(U\otimes id_{\frac{n}{k}}),\end{aligned}$$

Set $L = X^*V^*X(U\otimes id_{\frac{n}{k}})$, $N = Y^*V^*Y(U\otimes id_{\frac{n}{k}})$, then L and N commute with $(a\otimes id_{\frac{n}{k}})$, for all $a\in \mathrm{M}_k$, and

$$L^p = X^*(V^*)^pX(U^p\otimes id_{\frac{n}{k}}) = I, N^p = I.$$

Note that L, N commute with all $(a\otimes id_{\frac{n}{k}})$, then L, N belong to $I_k\otimes \mathrm{M}_{\frac{n}{k}}$. Let S and R be two unitaries in $I_k\otimes \mathrm{M}_{\frac{n}{k}}$ such that

$$\begin{aligned}SLS^* &= I_k\otimes diag(\lambda_1,...,\lambda_{\frac{n}{k}}),\\ RNR^* &= I_k\otimes diag(\mu_1,...,\mu_{\frac{n}{k}}).\end{aligned}$$

For $a_0 + a_1U_\rho + ... + a_{p-1}U_{\rho^{p-1}} \in \mathrm{M}_k\rtimes_{\alpha_k} Z_p$, take $a_0 = a_2 = ... = a_{p-1} = 0$, and $a_1 = U^*$, by Lemma 2.1, we have that

$$\begin{aligned}\tilde{\phi}(I, e^{i\frac{2\pi}{p}}I, ..., e^{i\frac{2(p-1)\pi}{p}}I) &= (X(U^*\otimes id_{\frac{n}{k}})X^*V, e^{\frac{i2\pi}{p}}X(U^*\otimes id_{\frac{n}{k}})X^*V, ...)\\ &= (XL^*X^*, e^{\frac{i2\pi}{p}}XL^*X^*, ...),\\ \tilde{\psi}(I, e^{i\frac{2\pi}{p}}I, ..., e^{i\frac{2(p-1)\pi}{p}}I) &= (Y(U^*\otimes id_{\frac{n}{k}})Y^*V, e^{\frac{i2\pi}{p}}Y(U^*\otimes id_{\frac{n}{k}})Y^*V, ...)\\ &= (YN^*Y^*, e^{\frac{i2\pi}{p}}YN^*Y^*, ...).\end{aligned}$$

Since $\tilde{\phi}_* = \tilde{\psi}_*$, then there exists a unitary Z such that $XL^*X^* = ZYN^*Y^*Z^*$, hence, $L = X^*ZYN^*Y^*Z^*X$, so $\{\lambda_1, ..., \lambda_{\frac{n}{k}}\} = \{\mu_1, ..., \mu_{\frac{n}{k}}\}$. Then there exists a unitary $\tilde{Z}\in I_k\otimes \mathrm{M}_{\frac{n}{k}}$ such that $L = \tilde{Z}N\tilde{Z}^*$. Hence,

$$X^*V^*X(U\otimes id_{\frac{n}{k}}) = \tilde{Z}Y^*V^*Y(U\otimes id_{\frac{n}{k}})\tilde{Z}^*,$$

which implies that $VX\tilde{Z}Y^* = X\tilde{Z}Y^*V$. Therefore $X\tilde{Z}Y^*\in B_n^{\beta_n}$, put $W = X\tilde{Z}Y^*$, then $\phi_k = AdW\circ\psi_n$.

(2). $A_k = \mathrm{M}_k, B_n = \underbrace{\mathrm{M}_n\oplus ...\oplus \mathrm{M}_n}_{p}$.

Let U be the order p unitary such that $\rho(a) = UaU^*$, $\forall a\in \mathrm{M}_k$. Let $X_1, ..., X_p$ be the untaries in M_n such that $\phi(a) = (X_1a\otimes id_{\frac{n}{k}}X_1^*, ..., X_pa\otimes id_{\frac{n}{k}}X_p^*)$; let $Y_1, ..., Y_p$ be the untaries in M_n such that $\psi(a) = (Y_1a\otimes id_{\frac{n}{k}}Y_1^*, ..., Y_pa\otimes id_{\frac{n}{k}}Y_p^*)$, $\forall a\in \mathrm{M}_k$. Since ϕ and ψ intertwines α_k and β_n, we obtains:

$$\begin{aligned}&(X_1(UaU^*\otimes id_{\frac{n}{k}})X_1^*, X_2(UaU^*\otimes id_{\frac{n}{k}})X_2^*, ..., X_p(UaU^*\otimes id_{\frac{n}{k}})X_p^*) =\\ &\qquad (X_pa\otimes id_{\frac{n}{k}}X_p^*, X_1a\otimes id_{\frac{n}{k}}X_1^*, ..., X_{p-1}a\otimes id_{\frac{n}{k}}X_{p-1}^*).\end{aligned}$$

Hence,

$$\begin{aligned}&X_1(UaU^*\otimes id_{\frac{n}{k}})X_1^* = X_pa\otimes id_{\frac{n}{k}}X_p^*,\\ &X_2(UaU^*\otimes id_{\frac{n}{k}})X_2^* = X_1a\otimes id_{\frac{n}{k}}X_1^*,\\ &, ...,\\ &X_p(UaU^*\otimes id_{\frac{n}{k}})X_p^* = X_{p-1}a\otimes id_{\frac{n}{k}}X_{p-1}^*.\end{aligned}$$

This implies that

$$\begin{aligned}
&X_p^*X_1U\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}X_p^*X_1U\otimes id_{\frac{n}{k}},\\
&X_1^*X_2U\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}X_1^*X_2U\otimes id_{\frac{n}{k}},\\
&,...,\\
&X_{p-1}^*X_pU\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}X_{p-1}^*X_pU\otimes id_{\frac{n}{k}}.
\end{aligned}$$

Similarly, we also have:

$$\begin{aligned}
&Y_p^*Y_1U\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}Y_p^*Y_1U\otimes id_{\frac{n}{k}},\\
&Y_1^*Y_2U\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}Y_1^*Y_2U\otimes id_{\frac{n}{k}},\\
&,...,\\
&Y_{p-1}^*Y_pU\otimes id_{\frac{n}{k}}a\otimes id_{\frac{n}{k}} = a\otimes id_{\frac{n}{k}}Y_{p-1}^*Y_pU\otimes id_{\frac{n}{k}}.
\end{aligned}$$

Our goal is to find a unitary $W=(W_1,...,W_p)\in B_n^{\beta_n}$, such that $\phi=AdW\circ\psi$. Note that $W\in B_n^{\beta_n}$ means that $(W_1,W_2,...,W_p)=(W_p,W_1,...,W_{p-1})$, namely, $W_1=W_2=...=W_p$.

Set

$$\begin{aligned}
&L_1=X_p^*X_1U\otimes id_{\frac{n}{k}}, N_1=Y_p^*Y_1U\otimes id_{\frac{n}{k}},\\
&,...,\\
&L_p=X_{p-1}^*X_pU\otimes id_{\frac{n}{k}}, N_p=Y_{p-1}^*Y_pU\otimes id_{\frac{n}{k}},
\end{aligned}$$

then by the calculation above, all of these L_i,N_i commute with $a\otimes id_{\frac{n}{k}}$, $\forall a\in \mathrm{M}_n$. Then $N_pL_p^*=Y_{p-1}^*Y_pX_p^*X_{p-1}$, which implies $X_pY_p^*=X_{p-1}L_pN_p^*Y_{p-1}^*$. Moreover, $X_{p-2}L_{p-1}L_pN_p^*N_{p-1}^*Y_{p-2}^*=X_{p-2}X_{p-2}^*X_{p-1}(U\otimes id_{\frac{n}{k}})L_pN_p^*(U^*\otimes id_{\frac{n}{k}})Y_{p-1}^*Y_{p-2}Y_{p-2}^*$ $=X_{p-1}L_pN_p^*Y_{p-1}^*$. Similarly, we have

$$\begin{aligned}
&X_{p-3}L_{p-2}L_{p-1}L_pN_p^*N_{p-1}^*N_{p-2}^*Y_{p-3}^*=X_{p-2}L_{p-1}L_pN_p^*N_{p-1}^*Y_{p-2}^*,\\
&,...,\\
&X_1L_2...L_pN_p^*...N_2^*Y_1^*=X_2L_3...L_pN_p^*...N_3^*Y_2^*.
\end{aligned}$$

So all of these terms equal to $X_{p-1}L_pN_p^*Y_{p-1}^*=X_pY_p^*$.

Put

$$W=(X_1L_2...L_pN_p^*...N_2^*Y_1^*, X_2L_3...L_pN_p^*...N_3^*Y_2^*,...,X_pY_p^*),$$

then $W\in B_n^{\beta_n}$, and $\phi_k=AdW\circ\psi_n$, since for each $i=1,...,p-1$, we have

$$X_iL_{i+1}...L_pN_p^*...N_{i+1}^*Y_i^*Y_i(a\otimes id_{\frac{n}{k}})Y_i^*Y_iN_{i+1}...N_pL_p^*...L_{i+1}^*X_i^*=X_i(a\otimes id_{\frac{n}{k}})X_i^*.$$

(3). $A_k=\underbrace{\mathrm{M}_k\oplus...\oplus\mathrm{M}_k}_{p}, B_n=\mathrm{M}_n$.

Let V be the unitary which implements the action of β_n, namely, $\rho(a)=VaV^*,\forall a\in\mathrm{M}_n$. Let X,Y be unitaries in M_n such that $\forall(a_1,a_2,...,a_p)\in\underbrace{\mathrm{M}_k\oplus...\oplus\mathrm{M}_k}_{p}$,

$$\begin{aligned}
&\phi(a_1,a_2,...,a_p)=X diag(a_1\otimes id_{\frac{n}{pk}},a_2\otimes id_{\frac{n}{pk}},...,a_p\otimes id_{\frac{n}{pk}})X^*,\\
&\psi(a_1,a_2,...,a_p)=Y diag(a_1\otimes id_{\frac{n}{pk}},a_2\otimes id_{\frac{n}{pk}},...,a_p\otimes id_{\frac{n}{pk}})Y^*.
\end{aligned}$$

This is the case since both ϕ_* and ψ_* intertwine the actions, and $\phi_*=\psi_*$.

Since ϕ and ψ intertwine the actions, we obtain that:

$$X\begin{pmatrix} a_p\otimes id_{\frac{n}{pk}} & & & \\ & a_1\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_{p-1}\otimes id_{\frac{n}{pk}} \end{pmatrix}X^*$$

$$=VX\begin{pmatrix} a_1\otimes id_{\frac{n}{pk}} & & & \\ & a_2\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p\otimes id_{\frac{n}{pk}} \end{pmatrix}X^*V^*.$$

and

$$Y\begin{pmatrix} a_p\otimes id_{\frac{n}{pk}} & & & \\ & a_1\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_{p-1}\otimes id_{\frac{n}{pk}} \end{pmatrix}Y^*$$

$$=VY\begin{pmatrix} a_1\otimes id_{\frac{n}{pk}} & & & \\ & a_2\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p\otimes id_{\frac{n}{pk}} \end{pmatrix}Y^*V^*.$$

Set $P=\begin{pmatrix} & & & I_{\frac{n}{k}} \\ I_{\frac{n}{k}} & & & \\ & \ddots & & \\ & & I_{\frac{n}{k}} & \end{pmatrix}$, then

$$P\begin{pmatrix} a_p\otimes id_{\frac{n}{pk}} & & & \\ & a_1\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_{p-1}\otimes id_{\frac{n}{pk}} \end{pmatrix}P^*$$

$$=\begin{pmatrix} a_1\otimes id_{\frac{n}{pk}} & & & \\ & a_2\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p\otimes id_{\frac{n}{pk}} \end{pmatrix}.$$

So we have:

$$XP^*P\begin{pmatrix} a_p\otimes id_{\frac{n}{pk}} & & & \\ & a_1\otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_{p-1}\otimes id_{\frac{n}{pk}} \end{pmatrix}P^*PX^*$$

$$= VX \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix} X^*V^*,$$

namely,

$$XP^* \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix} PX^*$$

$$= VX \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix} X^*V^*,$$

so

$$X^*V^*XP^* \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix} X^*V^*XP^*,$$

similarly,

$$Y^*V^*YP^* \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \otimes id_{\frac{n}{pk}} & & & \\ & a_2 \otimes id_{\frac{n}{pk}} & & \\ & & \ddots & \\ & & & a_p \otimes id_{\frac{n}{pk}} \end{pmatrix} Y^*V^*YP^*.$$

Put $L = X^*V^*XP^*$, $N = Y^*V^*YP^*$, then

$$L = diag(L_1, ..., L_p), N = diag(N_1, ..., N_p),$$

where each L_i and N_i belongs to $id_k \otimes \mathrm{M}_{\frac{n}{pk}}$, which means each of them commutes with any matrix in $\mathrm{M}_k \otimes id_{\frac{n}{pk}}$. Hence,

$$X^*V^*X = LP = \begin{pmatrix} & & & L_1 \\ L_2 & & & \\ & \ddots & & \\ & & L_p & \end{pmatrix},$$

and

$$Y^*V^*Y = NP = \begin{pmatrix} & & & N_1 \\ N_2 & & & \\ & \ddots & & \\ & & N_p & \end{pmatrix}.$$

Since V is an order p unitary, we obtain

$$\begin{pmatrix} & & & L_1 \\ L_2 & & & \\ & \ddots & & \\ & & L_p & \end{pmatrix}^p = I, \begin{pmatrix} & & & N_1 \\ N_2 & & & \\ & \ddots & & \\ & & N_p & \end{pmatrix}^p = I.$$

Form this we have that

$$\begin{aligned} &L_1L_p...L_2 = I, \\ &L_2L_1...L_3 = I, \\ &, ..., \\ &L_pL_{p-1}...L_1 = I, \end{aligned}$$

similarly,

$$\begin{aligned} &N_1N_p...N_2 = I, \\ &N_2N_1...N_3 = I, \\ &, ..., \\ &N_pN_{p-1}...N_1 = I. \end{aligned}$$

Set

$$Z = \begin{pmatrix} N_1L_p...L_2 & & & & \\ & N_2N_1L_p...L_3 & & & \\ & & \ddots & & \\ & & & N_{p-1}...N_1L_p & \\ & & & & I \end{pmatrix},$$

By using the relations above, we have that

$$Z\begin{pmatrix} & & & L_1 \\ L_2 & & & \\ & \ddots & & \\ & & L_p & \end{pmatrix}Z^* = \begin{pmatrix} & & & N_1 \\ N_2 & & & \\ & \ddots & & \\ & & N_p & \end{pmatrix},$$

namely, $ZX^*V^*XZ^* = Y^*V^*Y$. Put $W = XZ^*Y^*$, then $WV = VW$, so $W \in B_n^{\beta_n}$, and $\phi_k = AdW \circ \psi_n$.

(4). $A_k = \underbrace{\mathrm{M}_k \oplus ... \oplus \mathrm{M}_k}_{p}, B_n = \underbrace{\mathrm{M}_n \oplus ... \oplus \mathrm{M}_n}_{p}$.

Since ϕ intertwines the actions α_k and β_n, by calculation,

$$\phi_* = \begin{pmatrix} l_{11} & l_{12} & \dots & l_{1p} \\ l_{1p} & l_{11} & \dots & l_{1p-1} \\ \dots & \dots & \dots & \dots \\ l_{12} & l_{13} & \dots & l_{11} \end{pmatrix},$$

where $(l_{11}+l_{12}+...+l_{1p})k=n$, and ϕ is of the following form:

$$\phi(a_1,a_2,...,a_p)=(\phi_1(.),\phi_2(.),...,\phi_p(.)),\forall(a_1,a_2,...,a_p)\in\underbrace{\mathrm{M}_n\oplus...\oplus\mathrm{M}_n}_{p},$$

where (.) is the abbreviation of $(a_1,a_2,...,a_p)$, and

$$\phi_1(a_1,a_2,...,a_p)=Xdiag(a_1\otimes id_{l_{11}},a_2\otimes id_{l_{12}},...,a_p\otimes id_{l_{1p}})X^*,$$
$$\phi_1(a_1,a_2,...,a_p)=Xdiag(a_2\otimes id_{l_{11}},a_3\otimes id_{l_{12}},...,a_1\otimes id_{l_{1p}})X^*,$$
$$,...,$$
$$\phi_1(a_1,a_2,...,a_p)=Xdiag(a_p\otimes id_{l_{11}},a_1\otimes id_{l_{12}},...,a_{p-1}\otimes id_{l_{1p}})X^*,$$

here X is a unitary in M_n. Since $\phi_*=\psi_*$, similarly,

$$\psi(a_1,a_2,...,a_p)=(\psi_1(.),\psi_2(.),...,\psi_p(.)),\forall(a_1,a_2,...,a_p)\in\underbrace{\mathrm{M}_n\oplus...\oplus\mathrm{M}_n}_{p},$$

where (.) is the abbreviation of $(a_1,a_2,...,a_p)$, and

$$\psi_1(a_1,a_2,...,a_p)=Ydiag(a_1\otimes id_{l_{11}},a_2\otimes id_{l_{12}},...,a_p\otimes id_{l_{1p}})Y^*,$$
$$\psi_1(a_1,a_2,...,a_p)=Ydiag(a_2\otimes id_{l_{11}},a_3\otimes id_{l_{12}},...,a_1\otimes id_{l_{1p}})Y^*,$$
$$,...,$$
$$\psi_1(a_1,a_2,...,a_p)=Ydiag(a_p\otimes id_{l_{11}},a_1\otimes id_{l_{12}},...,a_{p-1}\otimes id_{l_{1p}})Y^*,$$

here Y is a unitary in M_n.

Put $W=(XY^*,...,XY^*)\in B_n^{\beta_n}$, then it is clear that $\phi=AdW\circ\psi$. □

Corollary 4.2. *Let ϕ_k and ψ_k be two homomorphisms from the finite dimensional C*-dynamical system (A_k,α_k,Z_p) to (B_n,β_n,Z_p). Denote by $\widetilde{\phi}_k$ and $\widetilde{\psi}_k$ the morphisms from $A_k\rtimes_{\alpha_k}Z_p$ to $B_n\rtimes_{\beta_n}Z_p$ induced by ϕ_k and ψ_k, respectively. If $\phi_{k*}=\psi_{k*}$ and $\widetilde{\phi}_{k*}=\widetilde{\psi}_{k*}$, then there exists a unitary W in $B_n^{\beta_n}$, the fixed point subalgebra of B_n, such that $\phi_k=AdW\circ\psi_k$.*

5. CLASSIFICATION

In this section, we prove the classification **Theorem** 1.1 by using Elliott's intertwining arguments.

Proof. First of all, by standard argument, the K-theoretic invariants of the AF C*-dynamical systems can be lifted to finite stages, namely, by passing to subsequences and changing notation, we could obtain the following intertwinings:

$$\begin{tikzcd}
(\mathrm{K}_0(A_1),\alpha_{1_*}) \arrow[r] \arrow[d] & (\mathrm{K}_0(A_2),\alpha_{2_*}) \arrow[r] \arrow[d] & \cdots \arrow[r] & (\mathrm{K}_0(A),\alpha_*) \arrow[d, Leftrightarrow] \\
(\mathrm{K}_0(B_1),\beta_{1_*}) \arrow[r] \arrow[ur] & (\mathrm{K}_0(B_2),\beta_{2_*}) \arrow[r] \arrow[ur] & \cdots \arrow[r] & (\mathrm{K}_0(B),\beta_*)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathrm{K}_0(A_1\rtimes_{\alpha_1}Z_p),\hat{\alpha}_{1_*}) \arrow[r] \arrow[d] & (\mathrm{K}_0(A_2\rtimes_{\alpha_2}Z_p),\hat{\alpha}_{2_*}) \arrow[r] \arrow[d] & \cdots \arrow[r] & (\mathrm{K}_0(A\rtimes_{\alpha}Z_p),\hat{\alpha}_*) \arrow[d, Leftrightarrow] \\
(\mathrm{K}_0(B_1\rtimes_{\beta_1}Z_p),\hat{\beta}_{1_*}) \arrow[r] \arrow[ur] & (\mathrm{K}_0(B_2\rtimes_{\beta_2}Z_p),\hat{\beta}_{2_*}) \arrow[r] \arrow[ur] & \cdots \arrow[r] & (\mathrm{K}_0(B\rtimes_{\beta}Z_p),\hat{\beta}_*).
\end{tikzcd}$$

Second, we would like to make the two intertwinings above to be compatible. Note that we have that:

$$\begin{array}{ccccccc} \mathrm{K}_0(A_1) & \to & \mathrm{K}_0(A) & \cong & \mathrm{K}_0(B) & \leftarrow\cdots\leftarrow & \mathrm{K}_0(B_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}_0(A_1 \rtimes_{\alpha_1} Z_p) & \to & \mathrm{K}_0(A \rtimes_{\alpha} Z_p) & \cong & \mathrm{K}_0(B \rtimes_{\beta} Z_p) & \leftarrow\cdots\leftarrow & \mathrm{K}_0(B_1 \rtimes_{\beta} Z_p) \end{array},$$

Hence, there exists n, such that

$$\begin{tikzcd} \mathrm{K}_0(A_1) \arrow[r] \arrow[d] & \mathrm{K}_0(B_n) \arrow[d] \\ \mathrm{K}_0(A_1 \rtimes_{\alpha_1} Z_p) \arrow[r] & \mathrm{K}_0(B_n \rtimes_{\beta_n} Z_p) \end{tikzcd}$$

commutes. After reindexing, the two intertwinings above could satisfy the following commutative diagrams:

$$\begin{tikzcd} \mathrm{K}_0(A_n) \arrow[r] \arrow[d] & \mathrm{K}_0(B_n) \arrow[d] \\ \mathrm{K}_0(A_n \rtimes_{\alpha_n} Z_p) \arrow[r] & \mathrm{K}_0(B_n \rtimes_{\beta_n} Z_p), \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{K}_0(B_n) \arrow[r] \arrow[d] & \mathrm{K}_0(A_{n+1}) \arrow[d] \\ \mathrm{K}_0(B_n \rtimes_{\beta_n} Z_p) \arrow[r] & \mathrm{K}_0(A_{n+1} \rtimes_{\alpha_{n+1}} Z_p). \end{tikzcd}$$

Also, these intertwinings can preserve the special elements and the units.

Now, we can apply the existence and uniqueness results on finite stages. By Corollary 3.2, we can lift each morphism of the invariant to a morphism between the dynamical systems. By Corollary 4.2, we can correct each morphism by an inner morphism commuting with the actions, so we obtain an intertwining of the dynamical systems:

$$\begin{tikzcd} (A_1, \alpha_1) \arrow[r] \arrow[d] & (A_2, \alpha_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & (A, \alpha) \arrow[d, dashed, "\psi"] \\ (B_1, \beta_1) \arrow[r] \arrow[ur] & (B_2, \beta_2) \arrow[r] \arrow[ur] & \cdots \arrow[r] & (B, \beta). \end{tikzcd}$$

Hence, (A, α) and (B, β) are isomorphic by an isomorphism ψ which induces F and ϕ. □

Acknowledgements. This paper was completed while the first author was a post-doctoral fellow at the Fields Institute. He is very grateful to Professor George A. Elliott for his support and to the Fields Institute for its hospitality. The second author is supported by a Ph.D fellowship from Hebei Normal University, China.

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